Subtlety of T^*_{feq}

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Abstract

In this note, we show that the theory T^*_{feq} is NTP_1, or *subtle*, following the proof of Theorem 2.1 in [10]. [1] Essential to the proof is the fact that a tree witnessing SOP_2 may be assumed to be highly indiscernible; this is Proposition 2.3 from [5]. We introduce the relevant types of tree properties and "tree indiscernibility," and give some basic facts before turning our attention to the theory T^*_{feq}. We note that T^*_{feq} is $\aleph_0$-categorical and has quantifier elimination (both of which are quite useful in the proof of subtlety), and that it has TP_2 (and therefore is not simple).[2]

1 Preliminaries

We begin by recalling the definition of the tree property.

Definition 1.1 ([9], Definition 0.1)**.**

1. A theory T has the tree property if there are a formula $\varphi(x; y)$, $k < \omega$, and sequences $a_\eta \in \mathfrak{C}$ ($\eta \in {}^{<\omega}\omega$) such that:
   - for any $\eta \in {}^{<\omega}\omega$, $\{\varphi(x, a_{\eta^\frown l}) : l < \omega\}$ is k-inconsistent, but
   - for every $\beta \in {}^{\omega}\omega$, $\{\varphi(x, a_{\beta\restriction n}) : n < \omega\}$ is consistent.
2. A theory T is *simple* if it does not have the tree property.

In [8], Shelah proves that if a theory has the tree property, then it has one of two "extreme" tree properties. The tree property specifies that instances of $\varphi(x; y)$ along a branch are consistent, while instances at nodes that are siblings are inconsistent. It leaves open the consistency of instances at incomparable, non-sibling nodes. It is therefore natural to consider tree properties that do make a decision on the consistency of these instances.

Definition 1.2.

1. A theory T has the *k-tree property of the first kind* ($k-\text{TP}_1$) if there are a formula $\varphi(x; y)$ and tuples $\{a_\alpha : \alpha \in {}^{<\omega}\omega\}$ such that:
   - for $\alpha_0, \ldots, \alpha_{k-1} \in {}^{<\omega}\omega$ pairwise incomparable, $\{\varphi(x, a_{\alpha_i}) : 1 \leq i \leq k\}$ is inconsistent, but

[1] The stated result in [10] is, in fact, that T^*_{feq} does not have SOP_1. The proof, however, relies on a faulty result from [2] (Claims 2.11 and 2.14). Those Claims were fixed in [3], a few years after [10] was published. The issue is, in short, that one cannot assume as great a degree of indiscernibility for SOP_1 trees as was initially thought. However, given that one *can* assume that SOP_2 trees are highly indiscernible, Shelah and Usvyatsov's argument for Theorem 2.1 shows that T^*_{feq} does not have SOP_2. It is that argument that we give here, filling in the details that were less obvious to us than they would have been to the authors of [10]. It is still conjectured that T^*_{feq} does not have SOP_1.

[2] This note is almost directly lifted from chapters 1.5 and 2 of the author's Ph.D. thesis.

- for $\beta \in {}^{\omega}\omega$, $\{\varphi(x, a_{\beta\restriction n}) : n \in \omega\}$ is consistent.

2. A theory has *TP_1* if it has 2-TP_1.

3. A theory is called *NTP_1* or *subtle* if it does not have TP_1.

Definition 1.3. A theory T has the *tree property of the second kind* (TP_2) if there are a formula $\varphi(x; y)$, tuples $\{a_\alpha : \alpha \in {}^{<\omega}\omega\}$, and $k < \omega$ such that:

- for any $\alpha \in {}^{<\omega}\omega$, $\{\varphi(x; a_{\alpha^\frown i}) : i < \omega\}$ is k-inconsistent, but
- for any n and any $\alpha_0, \ldots, \alpha_{n-1} \in {}^{<\omega}\omega$, no two of which are siblings, $\{\varphi(x; a_{\alpha_i}) : i < n\}$ is consistent.

A theory without TP_2 is called *NTP_2*.

Remark 1.4. TP_2 is equivalent to the following condition (which is more commonly given as the definition of TP_2): there are a formula $\varphi(x; y)$ and tuples $\{a_{i,j} : i, j < \omega\}$ such that

- for any $f : \omega \to \omega$, $\{\varphi(x; a_{i,f(i)}) : i < \omega\}$ is consistent, but
- for all $i < \omega$, $\{\varphi(x; a_{i,j}) : j < \omega\}$ is k-inconsistent for some k.

Various theories of valued fields - for example, any theory of an ultraproduct of p-adics - are NTP_2 but not simple [1].

Theorem 1.5 ([8], Theorem III.7.11)**.** *If a theory T has the tree property, then either T has TP_1 or T has TP_2.*

In [2], Dzamonja and Shelah introduce the properties SOP_1 and SOP_2:

Definition 1.6 ([2], Definition 2.2)**.**

1. T has SOP_2 if there are a formula $\varphi(x, y)$ and tuples a_η for $\eta \in {}^{<\omega}2$ such that
   - for every $\rho \in {}^{\omega}2$, the set $\{\varphi(x, a_{\rho\restriction n}) : n < \omega\}$ is consistent, while
   - if $\eta, \nu \in {}^{<\omega}2$ are incomparable, $\{\varphi(x, a_\eta), \varphi(x, a_\nu)\}$ is inconsistent.

2. T has SOP_1 if there are a formula $\varphi(x, y)$ and tuples a_η for $\eta \in {}^{<\omega}2$ such that
   - for $\rho \in {}^{\omega}2$ the set $\{\varphi(x, a_{\rho\restriction n}) : n < \omega\}$ is consistent, but
   - if $\nu^\frown\langle 0\rangle \trianglelefteq \eta \in {}^{<\omega}2$, then $\{\varphi(x, a_\eta), \varphi(x, a_{\nu^\frown\langle 1\rangle})\}$ is inconsistent.

Fact 1.7. *A theory T has TP_1 if and only if it has SOP_2. (See, e.g., [5].)*

We know that $SOP_2 \Rightarrow SOP_1 \Rightarrow$ TP. It is unknown whether SOP_2 and SOP_1 are equivalent.

2 Tree indiscernibility

In general, it is not easy to prove directly that a theory does not have a tree property (of any kind). The problem becomes somewhat more tractable if we may assume certain facts about the tree. In particular, it is helpful to assume that the tree in question has some level of *indiscernibility.* There are several kinds of tree indiscernibility, but the idea behind all of them is that given two configurations of nodes on the tree that "look alike," the parameters decorating the nodes of those two configurations should have the same type. The differences among the various notions of tree indiscernibility come from the different interpretations of what it means for two tree configurations to "look alike." (For comparison, in the case of an indiscernible *sequence*, two sets of elements from the sequence "look alike" if they have the same order type.) For an in-depth discussion of "generalized indiscernibles," see [7].

Remark 2.1 (Notation).

1. In this section, we will make use of overlines to denote tuples of nodes, and tuples of the tuples decorating those nodes. For example, $\overline{\eta}$ denotes, for some $n < \omega$, a sequence $(\eta_0, \ldots, \eta_{n-1})$, and $\overline{a}_{\overline{\eta}}$ denotes $(a_{\eta_0}, \ldots, a_{\eta_{n-1}})$. Note that each a_{η_i} may itself be a tuple, but we shall follow the same convention we use elsewhere, and not use an overline in this case. For $\overline{\eta} = (\eta_0, \ldots, \eta_{n-1})$ and $i < n$, we write $\eta_i \in \overline{\eta}$.

2. For a single node η, we will use $|\eta|$ to refer to the height of the node. That is, if η is a node in a q-branching tree, $|\eta| = k$ if $\eta \in {}^k q$.

The following definition is taken from [5], although an equivalent version of parts 1 through 3 appeared in [2] several years earlier. As the authors of [5] note, the terminology of part 4 of this definition is due to Scow [7].

Definition 2.2.

1. A tuple $\overline{\eta} \in {}^{<\omega} q$ is $\cap$-*closed* if for each $\eta_i, \eta_j \in \overline{\eta}$, $\eta_i \cap \eta_j \in \overline{\eta}$, too.

2. Given tuples $\overline{\eta}, \overline{\nu} \in {}^{<\omega} q$, $\overline{\eta} \approx_1 \overline{\nu}$ if:

   - $\overline{\eta}$ and $\overline{\nu}$ are $\cap$-closed tuples of the same arity, and
   - $\forall i, j < \lg(\overline{\eta})$ and $\forall t < q$,
     - $\eta_i \trianglelefteq \eta_j$ iff $\nu_i \trianglelefteq \nu_j$, and
     - $\eta_i^\frown \langle t \rangle \trianglelefteq \eta_j$ iff $\nu_i^\frown \langle t \rangle \trianglelefteq \nu_j$.

   We shall read $\overline{\eta} \approx_1 \overline{\nu}$ as "$\overline{\eta}$ is 1-similar to $\overline{\nu}$."

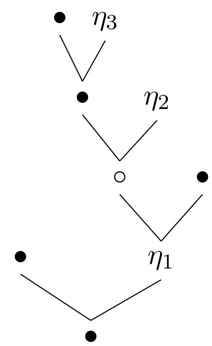
3. We say $\langle a_\eta | \eta \in {}^{<\omega} q \rangle$ is 1-*fti* (or 1-*fully tree indiscernible*) if for all $\overline{\eta}, \overline{\nu} \in {}^{<\omega} q$, $\overline{\eta} \approx_1 \overline{\nu}$ implies $\overline{a}_{\overline{\eta}} \equiv \overline{a}_{\overline{\nu}}$.

4. We say a sequence $\langle a_\eta | \eta \in {}^{<\omega} q \rangle$ is 1-*modeled* by a sequence $\langle b_\eta | \eta \in {}^{<\omega} q \rangle$ if, for any $d < \omega$, finite set $\Delta(x_0, \ldots, x_{d-1})$ of $\mathcal{L}$-formulae, and $\cap$-closed tuple $\overline{\eta} = \langle \eta_0, \ldots, \eta_{d-1} \rangle \in {}^{<\omega} q$, there exists $\overline{\nu} \in {}^{<\omega} q$ such that $\overline{\eta} \approx_1 \overline{\nu}$ and $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$.

Remark 2.3. The concept of 1-modeling allows us to "transform" one tree into another tree that is 1-fti (see Kim and Kim's Theorem 2.6, below). Importantly, this transformation preserves the property of being SOP_2 (Remark 2.7, below), so if we are given an SOP_2 tree, we can find a 1-fti SOP_2 tree.

Before stating the results that will be important in Chapter 2, we give a few examples of the properties defined above.

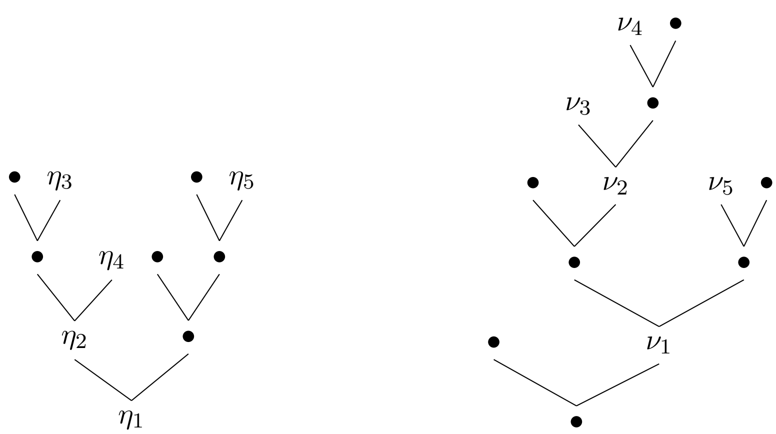
Example 2.4. Let $\overline{\eta} = (\langle 1\rangle, \langle 101\rangle, \langle 1001\rangle)$. The tuple $\overline{\eta}$ is not $\cap$-closed, because $\langle 101\rangle \cap \langle 1001\rangle = \langle 10\rangle \notin \overline{\eta}$.



Example 2.5. Let

$$\overline{\eta} = (\langle\rangle, \langle 0\rangle, \langle 001\rangle, \langle 01\rangle, \langle 111\rangle)$$
$$\overline{\nu} = (\langle 1\rangle, \langle 101\rangle, \langle 1010\rangle, \langle 10110\rangle, \langle 110\rangle)$$

Then $\overline{\eta} \approx_1 \overline{\nu}$.



Theorem 2.6 ([5], Proposition 2.3)**.** *Any sequence $\langle \alpha_\eta | \eta \in {}^{<\omega}q\rangle$ can be* 1*-modeled (in a sufficiently saturated model) by some* 1*-fti sequence $\langle b_\eta | \eta \in {}^{<\omega}q\rangle$.*

Remark 2.7 (See also [6], Remark 5.2, for the ω-branching case). If $\langle a_\eta | \eta \in {}^{<\omega}2\rangle$ witnesses SOP_2 for a formula $\psi(x, y)$ and $\langle b_\eta | \eta \in {}^{<\omega}2\rangle$ 1-models $\langle a_\eta | \eta \in {}^{<\omega}2\rangle$ in some model $\mathcal{M}$, then $\langle b_\eta | \eta \in {}^{<\omega}2\rangle$ is also an SOP_2 tree for $\psi(x, y)$.

Proof. 1. First we show that for any $\beta \in {}^{\omega}2$, $\{\psi(x, b_{\beta\restriction m}) | m \in \omega\}$ is consistent. Fix such a β, and consider a finite subset of $\{\psi(x, b_{\beta\restriction m}) | m \in \omega\}$. It is contained in a set of the form $\{\psi(x, b_{\beta\restriction m}) | m < n\}$ for some $n \in \omega$, so it suffices to show that sets of this form are consistent. Choose $n \in \omega$, and let $\overline{\eta} = \langle \beta \restriction 0, \ldots, \beta \restriction n-1\rangle$. This is an $\cap$-closed tuple. Consider $\Delta(y) := \{\exists x (\bigwedge_{i<n} \psi(x, y_i))\}$. By hypothesis, we can find a $\overline{\nu} \in {}^{<\omega}2$ such that $\overline{\eta} \approx_1 \overline{\nu}$ and $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$. Since $\overline{\nu} \approx_1 \overline{\eta}$, we have $\nu_i \trianglelefteq \nu_j$ for all $i \leq j$. That is, $\overline{\nu}$ lies on a branch. Since $\langle \overline{a}_\eta | \eta \in {}^{<\omega}2\rangle$ is an SOP_2 tree for ψ, $\{\psi(x, a_{\nu_i}) | i < n\}$ is consistent, and thus $\mathcal{M} \vDash \exists x(\bigwedge_{i<n} \psi(x, a_{\nu_i}))$. It follows that $\mathcal{M} \vDash \exists x(\bigwedge_{i<n} \psi(x, b_{\eta_i}))$, and thus the set $\{\psi(x, b_{\beta\restriction m}) | m < n\}$ is consistent, as desired.

2. Now suppose that $\alpha, \gamma \in {}^{<\omega}2$ are incomparable - we must show that $\{\psi(x, b_\alpha), \psi(x, b_\gamma)\}$ is inconsistent. Let $\overline{\eta} := \langle \alpha, \gamma, \alpha \cap \gamma\rangle$. This is clearly $\cap$-closed. Let

$$\Delta(y) := \{\exists x(\psi(x, y_0) \wedge \psi(x, y_1))\}.$$

We can find $\overline{\nu} \in {}^{<\omega}2$ such that $\overline{\nu} \approx_1 \overline{\eta}$ and $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$. Since $\alpha \cap \gamma$ is the most recent common ancestor of α and γ, it must be that one of α, γ is a descendant of $(\alpha \cap \gamma)^\frown\langle 0\rangle$, while the other is a descendant of $(\alpha \cap \gamma)^\frown\langle 1\rangle$. Without loss of generality, assume $(\alpha \cap \gamma)^\frown\langle 0\rangle \trianglelefteq \alpha$. It follows that $\nu_2^\frown\langle 0\rangle \trianglelefteq \nu_0$, while $\nu_2^\frown\langle 1\rangle \trianglelefteq \nu_1$, and thus, that ν_0 and ν_1 are incomparable. Since $\langle a_\eta | \eta \in {}^{<\omega}2\rangle$ is an SOP_2 tree for ψ, it follows that $\{\psi(x, a_{\nu_0}), \psi(x, a_{\nu_1})\}$ is inconsistent, and thus that $\mathcal{M} \vDash \neg\exists x(\psi(x, a_{\nu_0}) \wedge \psi(x, a_{\nu_1}))$. Since $\overline{b}_{\overline{\eta}} \equiv_\Delta \overline{a}_{\overline{\nu}}$, $\mathcal{M} \vDash \neg\exists x(\psi(x, b_\alpha) \wedge \psi(x, b_\gamma))$, and $\{\psi(x, b_\alpha), \psi(x, b_\gamma)\}$ is inconsistent, as desired.

□

Remark 2.8. 1-modeling does not necessarily preserve TP, however. We omit a formal argument, and simply observe that in a tree 1-modeled on a tree with TP, nodes that are siblings might be modeled on nodes that are not siblings in the original tree. Since TP only guarantees inconsistency among instances of the formula at sibling nodes, the new tree might not satisfy the inconsistency requirement.

The above facts tell us that if a theory T has TP_1 (and hence, SOP_2), then there are a formula $\psi(x; y)$ and a sequence $\langle a_\alpha | \alpha \in {}^{<\omega}2\rangle$ which is a 1-fti tree witnessing SOP_2 for $\psi(x; y)$. We would also like to be able to narrow down the set of formulae which we must show do not have SOP_2 in order to show that *no* formula has SOP_2. To that end, we show that we can "eliminate the disjunction" - that is, if a disjunction of formulae has SOP_2, then one of the disjuncts has SOP_2.

Lemma 2.9. *If $\langle a_\alpha : \alpha \in {}^{<\omega}2\rangle$ is a 1-fti tree witnessing SOP_2 for $\psi_0 \vee \psi_1$, then we can find a tree witnessing SOP_2 for either ψ_0 or ψ_1.*

Proof. We may assume that both $\psi_0(x, a_{\langle\rangle})$ and $\psi_1(x, a_{\langle\rangle})$ are consistent - otherwise $\langle a_\alpha : \alpha \in {}^{<\omega}2\rangle$ is already a tree for one or the other. (This uses the fact that the tree is 1-fti: if, say, $\psi_0(x, a_{\langle\rangle})$ is inconsistent, then so is $\psi_0(x, a_\alpha)$ for every $\alpha \in {}^{<\omega}2$. It follows that consistency of the instances of $\psi_0 \vee \psi_1$ along a branch implies consistency of the corresponding instances of ψ_1. Inconsistency of incomparable instances of the disjunction automatically implies inconsistency of incomparable instances of ψ_1, so the original tree witnesses SOP_2 for ψ_1.) Consider the leftmost branch (i.e. $\{\langle\rangle, \langle 0\rangle, \langle 00\rangle, \langle 000\rangle, \ldots\}$). Let b realize

$$\{(\psi_0 \vee \psi_1)(x; a_{\langle 0^n\rangle}) : n < \omega\},$$

where $\langle 0^0\rangle = \langle\rangle$, $\langle 0^1\rangle = \langle 0\rangle$, $\langle 0^2\rangle = \langle 00\rangle$, etc. Let f be a function from ω to $\{0, 1\}$ defined as follows:

$$f(j) = \begin{cases} 0 & \text{if } \vDash \psi_0(b; a_{\langle 0^j\rangle}) \\ 1 & \text{otherwise} \end{cases}$$

The function f must take at least one of $\{0,1\}$ as its value for arbitrarily large j. We may assume $f(j) = 0$ for arbitrarily large j. Now, for any $m \in \omega$, consider $\overline{\eta} := \{\langle\rangle, \langle 0\rangle, \ldots \langle 0^m\rangle\}$. By our assumption, we can find $j_0, \ldots, j_m \in \omega$ with $j_0 < \ldots < j_m$ and such that

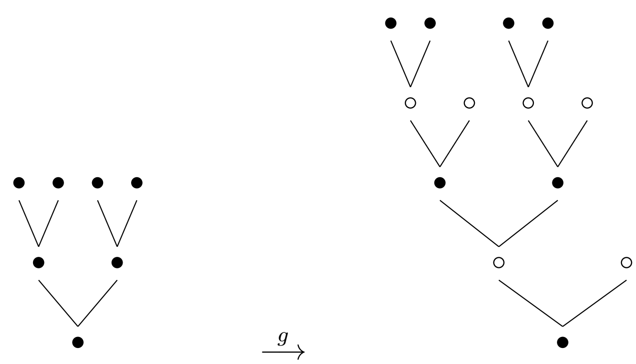
$$\{\psi_0(x, a_{\langle 0^{j_i}\rangle}) : i \leq m\}$$

is consistent. Let $\overline{\nu} := \{\langle 0^{j_0}\rangle, \langle 0^{j_1}\rangle, \ldots, \langle 0^{j_m}\rangle\}$. Clearly, $\overline{\eta} \approx_1 \overline{\nu}$. Since our tree is 1-fti, it follows that $\{\psi_0(x, a_{\langle 0^i\rangle}) : i \leq m\}$ is consistent. By compactness, $\{\psi_0(x, a_{\langle 0^i\rangle}) : i \in \omega\}$ is consistent.

We now define a function $g : {}^{<\omega}2 \to {}^{<\omega}2$ by induction, as follows:

$$g(\langle\rangle) := \langle\rangle$$
$$g(\alpha^\frown\langle 0\rangle) := g(\alpha)^\frown\langle 00\rangle$$
$$g(\alpha^\frown\langle 1\rangle) := g(\alpha)^\frown\langle 01\rangle$$

This is our pruning function: it picks out the nodes from the original tree that will form our new tree. We illustrate the first few steps of the function below. Think of the tree on the left as the tree we are constructing (the candidate for an SOP_2 tree for $\psi_0(x; y)$), and the tree on the right as the original tree. The black dots are the nodes in the image of g, and these are the nodes that form our new tree. The circles are nodes not in the image of g.



Notice that we are picking nodes from every other level of the original tree, and each node that appears in the image of g lies above the left child of the previous node appearing in the image. This construction gives us a tree all of whose branches "look like" (are 1-similar to) the leftmost branch of the original tree. We shall show that the new tree witnesses that $\psi_0(x; y)$ has SOP_2.

For $\alpha \in {}^{<\omega}2$, we let $b_\alpha := a_{g(\alpha)}$. We claim that $\langle b_\alpha : \alpha \in {}^{<\omega}2\rangle$ is a tree witnessing SOP_2 for ψ_0.

Claim 2.10. *Given $\eta, \nu \in {}^{<\omega}2$, $\eta \trianglelefteq \nu$ if and only if $g(\eta) \trianglelefteq g(\nu)$. (In particular, if η and ν are incomparable, so are $g(\eta)$ and $g(\nu)$.)*

Proof of claim. $\to$: Suppose $\eta \trianglelefteq \nu$. We show that $g(\eta) \trianglelefteq g(\nu)$ by induction on the distance between η and ν.

- Distance $= 0$; i.e. $\eta = \nu$. Then $g(\eta) = g(\nu)$, and so $g(\eta) \trianglelefteq g(\nu)$.
- Distance $= n+1$; i.e. $\nu = \alpha^\frown\langle i\rangle$ for some $i \in \{0,1\}$ and $\alpha \trianglerighteq \eta$, with the distance between η and α equal to n. By the induction hypothesis, $g(\eta) \trianglelefteq g(\alpha)$. By the definition of g,

  $$g(\nu) = g(\alpha^\frown\langle i\rangle) = g(\alpha)^\frown\langle 0i\rangle \trianglerighteq g(\alpha) \trianglerighteq g(\eta),$$

  as desired.

$\leftarrow$: We prove the contrapositive. Suppose $\eta \not\trianglelefteq \nu$. There are two cases.

- $\nu \lhd \eta$. Then there is $i \in \{0,1\}$ such that $\nu^\frown\langle i\rangle \trianglelefteq \eta$. By the forward direction of this claim, $g(\nu^\frown\langle i\rangle) \trianglelefteq g(\eta)$, hence $g(\nu)^\frown\langle 0i\rangle \trianglelefteq g(\eta)$. It follows that $g(\eta) \not\trianglelefteq g(\nu)$, as desired.
- ν and η are incomparable. We may assume without loss of generality that:

$$(\nu \cap \eta)^\frown\langle 0\rangle \trianglelefteq \nu;$$
$$(\nu \cap \eta)^\frown\langle 1\rangle \trianglelefteq \eta.$$

  Again by the forward direction of the claim, we have:

$$g(\nu) \trianglerighteq g((\nu \cap \eta)^\frown\langle 0\rangle) = g(\nu \cap \eta)^\frown\langle 00\rangle = (g(\nu \cap \eta)^\frown\langle 0\rangle)^\frown\langle 0\rangle;$$
$$g(\eta) \trianglerighteq g((\nu \cap \eta)^\frown\langle 1\rangle) = g(\nu \cap \eta)^\frown\langle 01\rangle = (g(\nu \cap \eta)^\frown\langle 0\rangle)^\frown\langle 1\rangle.$$

  Hence $g(\eta)$ and $g(\nu)$ are incomparable, and $g(\eta) \not\trianglelefteq g(\nu)$, as desired.

□

Remark 2.11. In fact, if $\eta \lhd \nu$, then $g(\eta)^\frown\langle 0\rangle \trianglelefteq g(\nu)$.

Proof of remark. If $\eta \lhd \nu$, then $\eta^\frown\langle i\rangle \trianglelefteq \nu$, for $i = 0$ or $i = 1$. By Claim 2.10 and the definition of g,

$$g(\eta^\frown\langle i\rangle) = g(\eta)^\frown\langle 0i\rangle = (g(\eta)^\frown\langle 0\rangle)^\frown\langle i\rangle \trianglelefteq \nu$$

□

Claim 2.12. *If $\eta, \nu \in {}^{<\omega}2$ are incomparable, then $\psi_0(x, b_\eta) \wedge \psi_0(x, b_\nu)$ is inconsistent.*

Proof of claim. Recall that $b_\eta = a_{g(\eta)}$ and $b_\nu = a_{g(\nu)}$. By the above, $g(\eta)$ and $g(\nu)$ are incomparable, so $\psi_0(x, a_{g(\eta)}) \wedge \psi_0(x, a_{g(\nu)})$ - i.e. $\psi_0(x, b_\eta) \wedge \psi_0(x, b_\nu)$ - is inconsistent, since $(\psi_0 \vee \psi_1(x, a_{g(\eta)})) \wedge (\psi_0 \vee \psi_1(x, a_{g(\nu)}))$ is. □

Claim 2.13. *If $\beta \in {}^\omega 2$, $\{\psi_0(x, b_{\beta\restriction i}) : i \in \omega\}$ is consistent.*

Proof of claim. It suffices to show that for any $n \in \omega$, $\{\psi_0(x, b_{\beta\restriction i}) : i \leq n\}$ is consistent. By Remark 2.11, if $i < j$, then $g(\beta \restriction i)^\frown\langle 0\rangle \trianglelefteq g(\beta \restriction j)$. Let $\overline{\eta} := (g(\beta \restriction i) : i \leq n)$. We just observed that $\overline{\eta}$ lies on a branch, and so it is $\cap$-closed. Now let $\overline{\nu} := (\langle 0^i\rangle : i \leq n)$. This is also $\cap$-closed. Furthermore, for $i, j \leq n$,

$$\begin{aligned}\langle 0^i\rangle \trianglelefteq \langle 0^j\rangle &\text{ iff } \langle 0^i\rangle = \langle 0^j\rangle \text{ or } \langle 0^i\rangle^\frown\langle 0\rangle \trianglelefteq \langle 0^j\rangle \\ &\text{ iff } i \leq j \\ &\text{ iff } g(\beta \restriction i) = g(\beta \restriction j) \text{ or } g(\beta \restriction i)^\frown\langle 0\rangle \trianglelefteq g(\beta \restriction j) \\ &\text{ iff } g(\beta \restriction i) \trianglelefteq g(\beta \restriction j)\end{aligned}$$

It follows that $\overline{\eta} \approx_1 \overline{\nu}$. Since $\{\psi_0(x, a_{\langle 0^i\rangle}) : i \leq n\}$ is consistent and $\langle a_\alpha : \alpha \in {}^{<\omega}2\rangle$ is a 1-fti tree, $\{\psi_0(x, a_{g(\beta\restriction i)}) : i \leq n\} = \{\psi_0(x, b_{\beta\restriction i}) : i \leq n\}$ is also consistent, as desired. □

Claims 2.12 and 2.13 tell us that $\langle b_\alpha : \alpha \in {}^{<\omega}2\rangle$ is a tree witnessing SOP_2 for ψ_0, so the proof of the lemma is finished. □

Corollary 2.14. *If T is $\aleph_0$-categorical and T has SOP_2, then there is a complete formula (that is, a formula isolating a complete type) $\varphi(x; y)$ with SOP_2.*

Proof. Suppose $\psi(x; y)$ has SOP_2, and $\psi(x; y)$ is incomplete. By $\aleph_0$-categoricity of T, there are finitely many complete types in the variables of ψ. Of these, let $p_0, \ldots, p_{n-1}$ list the ones that contain ψ. Again by $\aleph_0$-categoricity, each p_i is isolated by some formula - say, ψ_i. Clearly, $\psi_i \vdash \psi$ for each i. Further, since any realization of ψ realizes one of $p_0, \ldots, p_{n-1}$, $\psi \equiv \bigvee_{i<n} \psi_i$. By Lemma 2.9 and induction on n, $\bigvee_{i<n} \psi_i$ has SOP_2 if and only if one of the ψ_i does. □

We now make an observation about 1-fti trees: if the parameters at two different nodes in such a tree share any elements, then those elements are common to the parameters at *all* nodes in the tree.

Lemma 2.15. *Suppose $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is a 1-fti tree, where the b_η are tuples of length n. If for some η, $\nu \in {}^{<\omega}2$ and some $i \leq n$, $b^i_\eta = b^i_\nu$, then for* all *$\rho \in {}^{<\omega}2$, $b^i_\rho = b^i_\eta$.*

Proof. We begin by considering the case where η and ν are siblings, and have the same i^{th} coordinate as each other and as their parent.

Claim 2.16. *If for some $\rho \in {}^{<\omega}2$, $b^i_\rho = b^i_{\rho^\frown\langle 0\rangle} = b^i_{\rho^\frown\langle 1\rangle}$, then for all $\xi \in {}^{<\omega}2$, $b^i_\xi = b^i_\rho$.*

Proof of claim. We split the proof into two cases, depending on whether or not ρ is the root of the tree.

- **Case 1:** $\rho = \langle\rangle$. For all $\xi \in {}^{<\omega}2$, either

$$\langle\rangle\xi \approx_1 \langle\rangle\langle 0\rangle$$

  or

$$\langle\rangle\xi \approx_1 \langle\rangle\langle 1\rangle$$

  Since both $b^i_{\langle\rangle} = b^i_{\langle 0\rangle}$ and $b^i_{\langle\rangle} = b^i_{\langle 1\rangle}$, tree-indiscernibility implies that $b^i_{\langle\rangle} = b^i_\xi$.

- **Case 2:** $\rho \neq \langle\rangle$. Then either $\langle 0\rangle \trianglelefteq \rho$ or $\langle 1\rangle \trianglelefteq \rho$, and hence either

$$\langle\rangle\rho \approx_1 \rho\rho^\frown\langle 0\rangle$$

  or

$$\langle\rangle\rho \approx_1 \rho\rho^\frown\langle 1\rangle.$$

  Since both $b^i_\rho = b^i_{\rho^\frown\langle 0\rangle}$ and $b^i_\rho = b^i_{\rho^\frown\langle 1\rangle}$, $b^i_{\langle\rangle} = b^i_\rho$ by tree-indiscernibility. Then, since

$$(\langle\rangle, \langle 0\rangle, \langle 1\rangle) \approx_1 (\rho, \rho^\frown\langle 0\rangle, \rho^\frown\langle 1\rangle),$$

  tree-indiscernibility implies that $b^i_{\langle 0\rangle} = b^i_{\langle\rangle} = b^i_{\langle 1\rangle}$, and we are now in case 1 (giving the result that for all $\xi \in {}^{<\omega}2$, $b^i_\xi = b^i_{\langle\rangle} = b^i_\rho$).

□

We split the proof of the lemma into two cases, depending on whether or not η and ν are comparable. In the following, refer to $b^i_\eta = b^i_\nu$ as b^i.

- **Case 1.** Suppose η and ν lie on the same branch. Without loss of generality, suppose $\eta^\frown\langle 0\rangle \trianglelefteq \nu$. Then for any $\nu' \trianglerighteq \eta^\frown\langle 0\rangle$, $\eta\nu \approx_1 \eta\nu'$, and so $b^i_{\nu'} = b^i_\eta = b^i$. In particular, $b^i_{\eta^\frown\langle 0\rangle} = b^i_{\eta^\frown\langle 00\rangle} = b^i_{\eta^\frown\langle 01\rangle} = b^i$. By Claim 2.16, we are done (take ρ to be $\eta^\frown\langle 0\rangle$).

- **Case 2.** Suppose η and ν are incomparable. Without loss of generality, suppose $(\eta\cap\nu)^\frown\langle 0\rangle \trianglelefteq \eta$ and $(\eta\cap\nu)^\frown\langle 1\rangle \trianglelefteq \nu$. Then for all $\eta' \trianglerighteq (\eta\cap\nu)^\frown\langle 0\rangle$, $b^i_{\eta'} = b^i_\nu = b^i$ (since $(\nu,\eta,\eta\cap\nu) \approx_1 (\nu,\eta',\eta\cap\nu)$), and for all $\nu' \trianglerighteq (\eta\cap\nu)^\frown\langle 1\rangle$, $b^i_{\nu'} = b^i_\eta = b^i$ (since $(\eta,\nu,\eta\cap\nu) \approx_1 (\eta,\nu',\eta\cap\nu)$). In particular,

$$\begin{aligned} b^i &= b^i_{(\eta\cap\nu)^\frown\langle 0\rangle} \\ &= b^i_{(\eta\cap\nu)^\frown\langle 00\rangle} \\ &= b^i_{(\eta\cap\nu)^\frown\langle 01\rangle} \end{aligned}$$

  We apply Claim 2.16 again, this time taking ρ to be $(\eta\cap\nu)^\frown\langle 0\rangle$.

□

Corollary 2.17. *Suppose $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is a 1-fti tree, where the b_η are tuples of length n. If for some $\eta,\nu \in {}^{<\omega}2$ $c \in b_\eta \cap b_\nu$ (that is, c is a coordinate in each tuple, though not necessarily the same coordinate), then for all $\rho \in {}^{<\omega}2$, $c \in b_\rho$.*

Proof. Take $i \neq j < n$ such that $c = b^i_\eta = b^j_\nu$. (If $i = j$, the statement reduces to Lemma 2.15.) We split the proof into two cases, depending on whether or not η and ν are comparable.

- **Case 1:** Suppose η and ν are comparable. Without loss of generality, we may assume that $\eta^\frown\langle 0\rangle \trianglelefteq \nu$. Then

$$\begin{aligned} (\eta,\nu) &\approx_1 (\eta^\frown\langle 0\rangle, \eta^\frown\langle 00\rangle) \\ &\approx_1 (\eta, \eta^\frown\langle 00\rangle) \end{aligned}$$

  Since $b^i_\eta = b^j_\nu$, we have

$$\begin{aligned} b^i_{\eta^\frown\langle 0\rangle} &= b^j_{\eta^\frown\langle 00\rangle} \\ b^i_\eta &= b^j_{\eta^\frown\langle 00\rangle} \end{aligned}$$

  Hence $b^i_\eta = b^i_{\eta^\frown\langle 0\rangle}$, and by Lemma 2.15, $c = b^i_\eta = b^i_\rho$ for all $\rho \in {}^{<\omega}2$.

- **Case 2:** Suppose η and ν are incomparable. Without loss of generality, we may assume that $(\eta\cap\nu)^\frown\langle 0\rangle \trianglelefteq \eta$ and $(\eta\cap\nu)^\frown\langle 1\rangle \trianglelefteq \nu$. It follows that

$$(\eta,\nu,\eta\cap\nu) \approx_1 ((\eta\cap\nu)^\frown\langle 0\rangle, \nu, \eta\cap\nu).$$

  Since $b^i_\eta = b^j_\nu$, $b^i_{(\eta\cap\nu)^\frown\langle 0\rangle} = b^j_\nu$, and we have $c = b^i_{(\eta\cap\nu)^\frown\langle 0\rangle} = b^i_\eta$. By Lemma 2.15, $c = b^i_\rho$ for all $\rho \in {}^{<\omega}2$.

□

Remark 2.18. If $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is a 1-fti tree (where the b_η are tuples of length n) and c is an element of the common intersection of $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ (that is, $c \in b_\rho$ for all $\rho \in {}^{<\omega}2$), then for any $\eta, \nu \in {}^{<\omega}2$, $\text{tp}(b_\eta/c) = \text{tp}(b_\nu/c)$.

Proof. By the proof of Corollary 2.17, there is some $i < n$ such that $c = b^i_\rho$, all $\rho \in {}^{<\omega}2$. Since $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is 1-fti, $\text{tp}(b_\eta/c) = \text{tp}(b_\eta/b^i_\eta) = \text{tp}(b_\nu/b^i_\nu) = \text{tp}(b_\nu/c)$ for all $\eta, \nu \in {}^{<\omega}2$. □

We can generalize Corollary 2.17 a bit, so that it applies to the intersections of the definable closures of tuples at distinct nodes on a 1-fti tree. We begin by showing that we can add elements of the definable closure (in a reasonable way) to a 1-fti tree, and the resulting tree will also be 1-fti.

Lemma 2.19. *If $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is a 1-fti tree (where the b_η are tuples of length n), $\varphi(x, y)$ is a formula such that $\mathcal{M} \models (\exists^{=1}x)\varphi(x, b_{\langle\rangle})$, and for each $\eta \in {}^{<\omega}2$ c_η is the unique realization of $\varphi(x, b_\eta)$, then the tree $\langle b_\eta c_\eta : \eta \in {}^{<\omega}2\rangle$ is 1-fti.*

Proof. First note that the statement of the lemma makes sense: if $\mathcal{M} \models (\exists^{=1}x)\varphi(x; b_{\langle\rangle})$, then $\mathcal{M} \models (\exists^{=1}x)\varphi(x; b_\eta)$ for all $\eta \in {}^{<\omega}2$, so the c_η's exist. We must show that if $\overline{\eta}$ and $\overline{\nu}$ are tuples from ${}^{<\omega}2$ such that $\overline{\eta} \approx_1 \overline{\nu}$, then $\overline{b}_{\overline{\eta}}\overline{c}_{\overline{\eta}} \equiv \overline{b}_{\overline{\nu}}\overline{c}_{\overline{\nu}}$. Let $\overline{\eta} \approx_1 \overline{\nu}$ with $m = lg(\overline{\eta}) = lg(\overline{\nu})$, and let ψ be any formula (to which we add dummy variables, if necessary). Suppose that $\mathcal{M} \models \psi(\overline{b}_{\overline{\eta}}, \overline{c}_{\overline{\eta}})$. Then

$$\mathcal{M} \models \exists\overline{x}(\psi(\overline{b}_{\overline{\eta}}, \overline{x}) \wedge \bigwedge_{i<m} \varphi(x_i, b_{\eta_i})).$$

Since $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is 1-fti,

$$\mathcal{M} \models \exists\overline{x}(\psi(\overline{b}_{\overline{\nu}}, \overline{x}) \wedge \bigwedge_{i<m} \varphi(x_i, b_{\nu_i})).$$

Since c_{ν_i} is the only realization of $\varphi(x_i, b_{\nu_i})$, $\overline{c}_{\overline{\nu}}$ is the only possible realization of $\psi(\overline{b}_{\overline{\nu}}, \overline{x}) \wedge \bigwedge_{i<m} \varphi(x_i, b_{\nu_i}))$. Since we know this formula has a realization, $\mathcal{M} \models \psi(\overline{b}_{\overline{\nu}}, \overline{c}_{\overline{\nu}})$. Since ψ was arbitrary, $\overline{b}_{\overline{\eta}}\overline{c}_{\overline{\eta}} \equiv \overline{b}_{\overline{\nu}}\overline{c}_{\overline{\nu}}$, and $\langle b_\eta c_\eta : \eta \in {}^{<\omega}2\rangle$ is 1-fti. □

Corollary 2.20. *If $\langle b_\eta : \eta \in {}^{<\omega}2\rangle$ is a 1-fti tree and for some η, ν ($\eta \neq \nu$)*

$$c \in \text{dcl}(b_\eta) \cap \text{dcl}(b_\nu),$$

then $c \in \text{dcl}(b_\rho)$ for all $\rho \in {}^{<\omega}2$.

Proof. Suppose φ_1 witnesses that $c \in \text{dcl}(b_\eta)$ and φ_2 witnesses that $c \in \text{dcl}(b_\nu)$. For each $\rho \in {}^{<\omega}2$, let c_ρ be the unique realization of $\varphi_1(x, b_\rho)$, and let d_ρ be the unique realization of $\varphi_2(x, b_\rho)$. (So $c_\eta = c = d_\nu$.) By Lemma 2.19, the tree $\langle b_\rho c_\rho d_\rho : \rho \in {}^{<\omega}2\rangle$ is 1-fti. Since $c \in b_\eta c_\eta d_\eta \cap b_\nu c_\nu d_\nu$, $c \in b_\rho c_\rho d_\rho$ - and hence, $c \in \text{dcl}(b_\rho)$ - for all $\rho \in {}^{<\omega}2$, by Corollary 2.17. (In fact, by the proof of Corollary 2.17, $c = c_\rho = d_\rho$ for all $\rho \in {}^{<\omega}2$.) □

3 The theory

We take $\mathcal{L}$ to be a language whose signature has two sorts: one, P, for points, and another, E, for equivalence relations, and one three-place relation, R, on $P \times P \times E$. Given an E-element r, and two P-elements, a and b, we shall also write $r(a, b)$ for $R(a, b, r)$. We shall use lowercase letters x, y, z to denote variables of sort P, uppercase letters X, Y, Z to denote variables of sort E, lowercase a, b, c, d to denote parameters of sort P, and lowercase r, s, t to denote parameters of sort E. The universal $\mathcal{L}$-theory T_0 says that for each $r \in E$, $r(\cdot,\cdot)$ is an equivalence relation on P:

$$T_0 := \{\forall X, y(X(y,y)), \forall X, y, z(X(y,z) \to X(z,y)),\\ \forall X, y, z, w((X(y,z) \wedge X(z,w)) \to X(y,w))\}$$

Fact 3.1. *The class $\mathcal{K}$ of finite models of T_0 has the Hereditary Property, the Joint Embedding Property, and the Amalgamation Property, and hence (see, e.g., [4], Theorem 6.1.2) has a Fraïssé limit, $\mathcal{F}$.*

Proof. We recall the definition of each property before proving that $\mathcal{K}$ has it.

1. **Hereditary Property**: If $\mathcal{A} \in \mathcal{K}$ and $\mathcal{B}$ is a finitely generated substructure of $\mathcal{A}$, then $\mathcal{B} \in \mathcal{K}$.

   (Note that as $\mathcal{K}$ consists of finite structures in a relational language, "finitely generated" just means "finite" here.) By universality of T_0, any substructure of a (finite) model of T_0 is itself a (finite) model of T_0.

2. **Joint Embedding Property**: If $\mathcal{A}, \mathcal{B} \in \mathcal{K}$, then there is $\mathcal{C} \in \mathcal{K}$ such that $\mathcal{A}$ and $\mathcal{B}$ are embeddable in $\mathcal{C}$.

   We shall construct $\mathcal{C}$. Suppose

   $$A = \{a_0, \ldots, a_{n-1}, r_0, \ldots, r_{m-1}\}, \text{ and}\\ B = \{b_0, \ldots, b_{n'-1}, s_0, \ldots, s_{m'-1}\}.$$

   Take the universe of $\mathcal{C}$ to be the disjoint union of the universes of $\mathcal{A}$ and $\mathcal{B}$, that is,

   $$C = \{a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n'-1}, r_0, \ldots, r_{m-1}, s_0, \ldots, s_{m'-1}\}$$

   where the a_i and b_i, and the r_i and s_i are all distinct. (We may replace $\mathcal{B}$ by an isomorphic copy, if necessary, to achieve this.) We now define $R(\cdot,\cdot,\cdot)$ on $\mathcal{C}$.

   - For $i < m$, $j, k < n$, and $j', k' < n'$,

     $$\mathcal{C} \models r_i(a_j, a_k) \text{ if and only if } \mathcal{A} \models r_i(a_j, a_k);\\ \mathcal{C} \models \neg r_i(a_j, b_{j'}) \wedge \neg r_i(b_{j'}, a_j);\\ \mathcal{C} \models r_i(b_{j'}, b_{k'}).$$

   - For $i' < m'$, $j, k < n$, and $j', k' < n'$,

     $$\mathcal{C} \models s_{i'}(b_{j'}, b_{k'}) \text{ if and only if } \mathcal{B} \models s_{i'}(b_{j'}, b_{k'});\\ \mathcal{C} \models \neg s_{i'}(b_{j'}, a_j) \wedge \neg s_{i'}(a_j, b_{j'});\\ \mathcal{C} \models s_{i'}(a_j, a_k).$$

   In $\mathcal{C}$, each r_i has all of the classes it has in $\mathcal{A}$, plus an additional class for the b_i's. Similarly, each s_i has all of the classes it has in $\mathcal{B}$, plus an additional class for the a_i's. It is clear that each of the r_i's and s_j's is an equivalence relation on the points of $\mathcal{C}$, and that the inclusion maps from $\mathcal{A}$ into $\mathcal{C}$ and from $\mathcal{B}$ into $\mathcal{C}$ are embeddings.

3. **Amalgamation Property**: If $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathcal{K}$ and $e : \mathcal{A} \to \mathcal{B}$, $f : \mathcal{A} \to \mathcal{C}$ are embeddings, then there are $\mathcal{D} \in \mathcal{K}$ and embeddings $g : \mathcal{B} \to \mathcal{D}$, $h : \mathcal{C} \to \mathcal{D}$ such that $ge = hf$.

   We construct $\mathcal{D}$. Let the universe of $\mathcal{D}$ be the disjoint union of $B \setminus e(A)$, $C \setminus f(A)$, and A, where

$$\begin{aligned} B \setminus e(A) &= \{b_0, \ldots, b_{n_B-1}, s_0, \ldots, s_{m_B-1}\}, \\ C \setminus f(A) &= \{c_0, \ldots, c_{n_C-1}, t_0, \ldots, t_{m_C-1}\}, \\ A &= \{a_0, \ldots, a_{n_A-1}, r_0, \ldots, r_{m_A-1}\} \end{aligned}$$

   Our definition of $R^{\mathcal{D}}$ is similar to that of $R^{\mathcal{C}}$ in the proof of JEP: each s_i has the same classes as in $\mathcal{B}$, plus an additional class for the elements of $C \setminus f(A)$, and each t_i has the same classes as in $\mathcal{C}$, plus an additional class for the elements of $B \setminus e(A)$. We must be more careful when extending the definition of the r_i's, however, in order to make sure they satisfy transitivity. Thus, in our formal definition of $R^{\mathcal{D}}$, we consider three separate cases, depending on whether the equivalence relation element is from A, $B \setminus e(A)$, or $C \setminus f(A)$.

   - For all $i < m_B$, $j_A, k_A < n_A$, $j_B, k_B < n_B$, and $j_C, k_C < n_C$,

$$\begin{aligned} &\mathcal{D} \models s_i(a_{j_A}, a_{k_A}) \text{ if and only if } \mathcal{B} \models s_i(e(a_{j_A}), e(a_{j_A})); \\ &\mathcal{D} \models s_i(b_{j_B}, b_{k_B}) \text{ if and only if } \mathcal{B} \models s_i(b_{j_B}, b_{k_B}); \\ &\mathcal{D} \models s_i(c_{j_C}, c_{k_C}); \\ &\mathcal{D} \models \begin{cases} s_i(a_{j_A}, b_{j_B}) \wedge s_i(b_{j_B}, a_{j_A}) & \text{if } \mathcal{B} \models s_i(e(a_{j_A}), b_{j_B}), \\ \neg s_i(a_{j_A}, b_{j_B}) \wedge \neg s_i(b_{j_B}, a_{j_A}) & \text{otherwise;} \end{cases} \\ &\mathcal{D} \models \neg s_i(a_{j_A}, c_{j_C}) \wedge \neg s_i(c_{j_C}, a_{j_A}) \wedge \neg s_i(b_{j_B}, c_{j_C}) \wedge \neg s_i(c_{j_C}, b_{j_B}). \end{aligned}$$

     We can see that for any s_i, elements of $\mathcal{B}$ are s_i-related in $\mathcal{D}$ if and only if they are s_i-related in $\mathcal{B}$, while elements of $\mathcal{C} \setminus f(A)$ are all s_i-related to one another, and are not s_i-related to any element of $\mathcal{B}$.

   - For all $i < m_C$, $j_A, k_A < n_A$, $j_B, k_B < n_B$, and $j_C, k_C < n_C$,

$$\begin{aligned} &\mathcal{D} \models t_i(a_{j_A}, a_{k_A}) \text{ if and only if } \mathcal{C} \models t_i(f(a_{j_A}), f(a_{k_A})); \\ &\mathcal{D} \models t_i(c_{j_C}, c_{k_C}) \text{ if and only if } \mathcal{C} \models t_i(c_{j_C}, c_{k_C}); \\ &\mathcal{D} \models t_i(b_{j_B}, b_{k_B}); \\ &\mathcal{D} \models \begin{cases} t_i(a_{j_A}, c_{j_C}) \wedge t_i(c_{j_C}, a_{j_A}) & \text{if } \mathcal{C} \models t_i(f(a_{j_A}), c_{j_C}), \\ \neg t_i(a_{j_A}, c_{j_C}) \wedge \neg t_i(c_{j_C}, a_{j_A}) & \text{otherwise;} \end{cases} \\ &\mathcal{D} \models \neg t_i(a_{j_A}, b_{j_B}) \wedge \neg t_i(b_{j_B}, a_{j_A}) \wedge \neg t_i(c_{j_C}, b_{j_B}) \wedge \neg t_i(b_{j_B}, c_{j_C}). \end{aligned}$$

     As in the definition of s_i on $\mathcal{D}$, elements of $\mathcal{C}$ are t_i-related in $\mathcal{D}$ if and only if they are t_i-related in $\mathcal{C}$, while the elements of $\mathcal{B} \setminus e(A)$ form a separate t_i-class.

- For all $i < m_A$, $j_A, k_A < n_A$, $j_B, k_B < n_B$, and $j_C, k_C < n_C$,

$$
\begin{aligned}
&\mathcal{D} \vDash r_i(a_{j_A}, a_{k_A}) \text{ if and only if } \mathcal{A} \vDash r_i(a_{j_A}, a_{k_A}) \\
&\quad \text{if and only if } \mathcal{B} \vDash e(r_i)(e(a_{j_A}), e(a_{k_A})) \\
&\quad \text{if and only if } \mathcal{C} \vDash f(r_i)(f(a_{j_A}), f(a_{k_A})) \\
&\mathcal{D} \vDash r_i(b_{j_B}, b_{k_B}) \text{ if and only if } \mathcal{B} \vDash e(r_i)(b_{j_B}, b_{k_B}) \\
&\mathcal{D} \vDash r_i(c_{j_C}, c_{k_C}) \text{ if and only if } \mathcal{C} \vDash f(r_i)(c_{j_C}, c_{k_C}) \\
&\mathcal{D} \vDash \begin{cases} r_i(a_{j_A}, b_{j_B}) \wedge r_i(b_{j_B}, a_{j_A}) & \text{if } \mathcal{B} \vDash e(r_i)(e(a_{j_A}), b_{j_B}), \\ \neg r_i(a_{j_A}, b_{j_B}) \wedge \neg r_i(b_{j_B}, a_{j_A}) & \text{otherwise;} \end{cases} \\
&\mathcal{D} \vDash \begin{cases} r_i(a_{j_A}, c_{j_C}) \wedge r_i(c_{j_C}, a_{j_A}) & \text{if } \mathcal{C} \vDash f(r_i)(f(a_{j_A}), c_{j_C}), \\ \neg r_i(a_{j_A}, c_{j_C}) \wedge \neg r_i(c_{j_C}, a_{j_A}) & \text{otherwise;} \end{cases} \\
&\mathcal{D} \vDash r_i(b_{j_B}, c_{j_C}) \wedge r_i(c_{j_C}, b_{j_B}) \text{ if there is } 1 \leq \ell \leq n_A \text{ such that} \\
&\qquad\qquad \mathcal{B} \vDash e(r_i)(b_{j_B}, e(a_\ell)) \text{ and} \\
&\qquad\qquad \mathcal{C} \vDash f(r_i)(f(a_\ell), c_{j_C}), \\
&\quad \text{otherwise, } \mathcal{D} \vDash \neg r_i(b_{j_B}, c_{j_C}) \wedge \neg r_i(c_{j_C}, b_{j_B}).
\end{aligned}
$$

  Once again, elements of $\mathcal{B}$ are r_i-related in $\mathcal{D}$ if and only if they are $e(r_i)$-related in $\mathcal{B}$, and elements of $\mathcal{C}$ are r_i-related in $\mathcal{D}$ if and only if they are $f(r_i)$-related in $\mathcal{C}$. We extend r_i to $(B \setminus e(A)) \times (C \setminus f(A))$ so that b_j and c_k are r_i related just in case there is some element of A that "connects" them. This is well defined: either b_j and c_k are $e(r_i)$-related (respectively, $f(r_i)$-related) to (images of) all the same elements of A, or to (images of) none of the same elements of A, since e and f preserve the r_i-classes of A.

It should be clear that each r_i, s_j, and t_k is an equivalence relation, so $\mathcal{D} \vDash T_0$. We define $g : \mathcal{B} \to \mathcal{D}$ and $h : \mathcal{C} \to \mathcal{D}$ in the obvious way: $g \restriction (B \setminus e(A))$ and $h \restriction (C \setminus f(A))$ are the identity map, while $g \restriction e(A) = e^{-1}$ and $h \restriction f(A) = f^{-1}$. The maps g and h are embeddings, and $ge = hf$.

□

Let T^*_{feq} be the theory of $\mathcal{F}$. As the theory of the Fraïssé limit of the finite models of T_0, T^*_{feq} is $\aleph_0$-categorical and has elimination of quantifiers (see [4], Theorem 6.4.1), and it is the model completion of T_0. For convenience, we spell out the intermediate theory T_{feq}:

$$
\begin{aligned}
T_{\text{feq}} := T_0 &\cup \left\{ \forall X \exists y_0, \ldots, y_{n-1} \left(\bigwedge_{i \neq j} \neg X(y_i, y_j) \right) : n < \omega \right\} \\
&\cup \left\{ \exists X_0, \ldots, X_{n-1} \left(\bigwedge_{i \neq j} X_i \neq X_j \right) : n < \omega \right\} \\
&\cup \left\{ \forall X_0, \ldots, X_{n-1} \forall y_0, \ldots, y_{n-1} \exists y \left(\bigwedge_{i \neq j} X_i \neq X_j \to \bigwedge_{i < n} X_i(y, y_i) \right) : n < \omega \right\}
\end{aligned}
$$

T_{feq} says that each equivalence relation has infinitely many classes, that there are infinitely many E-sort elements, and that given any n distinct classes $r_0, \ldots, r_{n-1}$, and n (not necessarily distinct) points $a_0, \ldots, a_{n-1}$, we can find a point that is r_i-equivalent to a_i for $i < n$. We shall refer to this as the "cross-cutting axiom."

Fact 3.2. *The theory T^*_{feq} is an extension of T_{feq}.*

Proof. Suppose $\mathcal{M} \models T^*_{\text{feq}}$. We show $\mathcal{M} \models T_{\text{feq}}$.

1. For each $n < \omega$, $\mathcal{M} \models \forall X \exists y_0, \ldots, y_{n-1} \left(\bigwedge_{i \neq j} \neg X(y_i, y_j) \right)$.

   Suppose $r \in E^{\mathcal{M}}$, and r has m classes for some $m < n$. We build a model $\mathcal{N}$ of T_0 extending $\mathcal{M}$: let the universe of $\mathcal{N}$ be $M \cup \{a_0, \ldots, a_{n-m-1}\}$ (where $a_0, \ldots, a_{n-m-1}$ are new elements of sort P). For $s \in E^{\mathcal{M}} \setminus \{r\}$, let the new elements form a single, new s-class. (This is not actually relevant, but we must make some choice with regard to the s-classes of the new elements.) Let each new element form its own r-class. More precisely, we define $R^{\mathcal{N}}$ by:

   $$\mathcal{N} \models s(b, c) \text{ if and only if } \mathcal{M} \models s(b, c) \text{ for all } s \in E^{\mathcal{M}}, b, c \in P^{\mathcal{M}};$$
   $$\mathcal{N} \models \neg s(b, a_i) \wedge \neg s(a_i, b) \text{ for any } s \in E^{\mathcal{M}}, b \in P^{\mathcal{M}} \text{ and } i < n - m;$$
   $$\mathcal{N} \models \bigwedge_{i,j < n-m} s(a_i, a_j) \text{ for all } s \in E^{\mathcal{M}} \setminus \{r\};$$
   $$\mathcal{N} \models \bigwedge_{i \neq j} \neg r(a_i, a_j).$$

   Under this definition of $R^{\mathcal{N}}$, $\mathcal{N} \models T_0$, $\mathcal{N}$ extends $\mathcal{M}$, and $\mathcal{N} \models \exists y_0, \ldots, y_{n-1} \left(\bigwedge_{i \neq j} \neg r(y_i, y_j) \right)$. As $\mathcal{M}$ is existentially closed for models of T_0, $\mathcal{M}$ must already contain at least n r-classes. Since r was arbitrary, we have that

   $$\mathcal{M} \models \forall X \exists y_0, \ldots, y_{n-1} \left(\bigwedge_{i \neq j} \neg X(y_i, y_j) \right),$$

   as desired.

2. For each $n < \omega$, $\mathcal{M} \models \exists X_0, \ldots, X_{n-1} \left(\bigwedge_{i \neq j} X_i \neq X_j \right)$.

   Again, we build a model $\mathcal{N}$ of T_0 extending $\mathcal{M}$. Let the universe of $\mathcal{N}$ be $\mathcal{M} \cup \{r_0, \ldots, r_{n-1}\}$, and define $R^{\mathcal{N}}$ as follows:

   $$\mathcal{N} \models s(a, b) \text{ if and only if } \mathcal{M} \models s(a, b) \text{ for all } s \in E^{\mathcal{M}},\ a, b \in P^{\mathcal{M}};$$
   $$\mathcal{N} \models r_i(a, b) \text{ for any } i < n, \text{ any } a, b \in P^{\mathcal{M}}.$$

That is, $\mathcal{N}$ adds n equivalence relations to $\mathcal{M}$, each of which has only one class. It is clear that $\mathcal{M}$ embeds into $\mathcal{N}$, $\mathcal{N} \models T_0$, and $\mathcal{N} \models \exists X_0, \dots, X_{n-1} \left(\bigwedge_{i \neq j} X_i \neq X_j \right)$. Since $\mathcal{M}$ is existentially closed for models of T_0, it satisfies the desired axiom.

3. For each $n < \omega$, $\mathcal{M} \models \forall X_0, \dots, X_{n-1} \forall y_0, \dots, y_{n-1} \exists y \left(\bigwedge_{i \neq j} X_i \neq X_j \to \bigwedge_{i<n} X_i(y, y_i) \right)$.

Suppose $r_0, \dots, r_{n-1}$ are distinct elements of $E^{\mathcal{M}}$, and $a_0, \dots, a_{n-1}$ are (not necessarily distinct) elements of $P^{\mathcal{M}}$. Again, we build a model $\mathcal{N}$ of T_0 extending $\mathcal{M}$. Let the universe of $\mathcal{N}$ be the union of M and a single new element, b. For each $s \in E^{\mathcal{M}} \setminus \{r_0, \dots, r_{n-1}\}$, let b be in a new s-class of its own. For each r_i, put b in the same r_i class as a_i (and all of the other elements in a_i's r_i-class, according to $\mathcal{M}$). More precisely, we define $R^{\mathcal{N}}$ as follows:

$$\mathcal{N} \models s(c,d) \text{ if and only if } \mathcal{M} \models s(c,d) \text{ for all } s \in E^{\mathcal{M}},\ c,d \in P^{\mathcal{M}};$$
$$\mathcal{N} \models \neg s(c,b) \wedge \neg s(b,c) \text{ for all } s \in E^{\mathcal{M}},\ c \in P^{\mathcal{M}};$$
$$\mathcal{N} \models \bigwedge_{i<n} r_i(b, a_i) \wedge r_i(a_i, b);$$
$$\mathcal{N} \models \begin{cases} r_i(b,c) \wedge r_i(c,b) & \text{if } \mathcal{M} \models r_i(a_i, c) \\ \neg r_i(b,c) \wedge \neg r_i(c,b) & \text{otherwise} \end{cases} \qquad \text{for any } c \in P^{\mathcal{M}}.$$

With this definition of $R^{\mathcal{N}}$, it is clear that $\mathcal{N} \models T_0$, $\mathcal{N}$ is an extension of $\mathcal{M}$, and $\mathcal{N} \models \exists y \left(\bigwedge_{i<n} r_i(y, a_i) \right)$. Again, we note that since $\mathcal{M}$ is existentially closed for models of T_0, this must already be true in $\mathcal{M}$, that is,

$$\mathcal{M} \models \exists y \left(\bigwedge_{i<n} r_i(y, a_i) \right).$$

Since $r_0, \dots, r_{n-1}$ were arbitrary distinct elements of $E^{\mathcal{M}}$ and $a_0, \dots, a_{n-1}$ were arbitrary elements of $P^{\mathcal{M}}$, $\mathcal{M}$ satisfies the desired axiom.

□

Fact 3.3. *If T extends T_{feq}, the formula $\varphi(x; y, Z) := Z(x, y)$ has the tree property of the second kind, and in particular, T is not simple.*

Proof. Work in any sufficiently saturated model $\mathcal{M}$ of T_{feq}. We begin by showing that the following type, in variables X_i for $i \in \omega$, $z_{(i,j)}$ for $i, j \in \omega$, and y_α for $\alpha \in {}^\omega\omega$ is consistent:

$$\begin{aligned} \pi(X_i, z_{(j,k)}, y_\alpha : i,j,k \in \omega, \alpha \in {}^\omega\omega) = & \bigcup_{\alpha \in {}^\omega\omega} \{X_i(y_\alpha, z_{(i,\alpha(i))}) : i < \omega\} \\ & \cup \{X_i \neq X_j : i < j < \omega\} \\ & \cup \bigcup_{i<\omega} \{\neg X_i(z_{(i,j)}, z_{(i,k)}) : j < k < \omega\}. \end{aligned}$$

Given a finite subset of this type, there are $n, m, l \in \omega$ such that the subset is contained in a type of the following form, in variables X_i for $i < n$, $y_{\alpha_0}, \dots, y_{\alpha_{l-1}}$ for $\alpha_i \in {}^\omega\omega$, and $z_{(i,j)}$ for $i < n$ and $j < m$, where $m > \max\{\alpha_j(i) : j < l, i < n\}$:

$$\begin{aligned}\pi_0(X_i, z_{(j,k)}, y_{\alpha_r} : i, j < n, k < m, r < l) = & \bigcup_{r<l} \{X_i(y_{\alpha_r}, z_{(i,\alpha_r(i))}) : i < n\} \\ & \cup \{X_i \neq X_j : i < j < n\} \\ & \cup \bigcup_{i<n} \{\neg X_i(z_{(i,j)}, z_{(i,k)}) : j < k < m\}.\end{aligned}$$

To realize π_0, we begin by choosing any n distinct equivalence relation elements $s_0, \dots, s_{n-1}$ from $\mathcal{M}$. We can do this because $E^{\mathcal{M}}$ is infinite. Then, for each i, we choose m elements $a_{(i,0)}, \dots, a_{(i,m-1)}$ of $P^{\mathcal{M}}$ from distinct s_i-classes. We can do this because each element of $E^{\mathcal{M}}$ has infinitely many classes. Lastly, for each $r < l$, we find b_{α_r} satisfying

$$\bigwedge_{i<n} s_i(y, a_{(i,\alpha_r(i))}).$$

Such b_{α_r} exist by the cross-cutting axiom of T_{feq}. By the choice of these elements,

$$\mathcal{M} \models \pi_0(s_i, a_{(j,k)}, b_{\alpha_r} : i, j < n, k < m, r < l).$$

By compactness, π is consistent and (by saturation) realized in $\mathcal{M}$. Let

$$(s_i, a_{(i,j)}, b_\alpha : i, j < \omega, \alpha \in {}^\omega\omega)$$

realize π in $\mathcal{M}$, and for each i, let $t_{(i,j)} = s_i$. To recap, we have:

$$\mathcal{M} \models t_{(i,\alpha(i))}(b_\alpha, a_{(i,\alpha(i))}) \tag{1}$$

for each $\alpha \in {}^\omega\omega$ and $i < \omega$, and

$$\mathcal{M} \models \neg\exists x (t_{(i,j)}(x, a_{(i,j)}) \wedge t_{(i,k)}(x, a_{(i,k)})) \tag{2}$$

for any $i < \omega$ and any $j \neq k < \omega$. (This last comes from the fact that $t_{(i,j)} = t_{(i,k)}$ and $a_{(i,j)}$ and $a_{(i,k)}$ are in different $t_{(i,j)}$-classes.) We claim that $(a_{(i,j)} t_{(i,j)} : i, j < \omega)$ witnesses that $\varphi(x; y, Z)$ has TP_2.

- For each $i < \omega$,
$$\{\varphi(x; a_{(i,j)}, t_{(i,j)}) : j < \omega\} = \{t_{(i,j)}(x, a_{(i,j)}) : j < \omega\}$$
is inconsistent, by 2, above.
- For each $\alpha \in {}^\omega\omega$,
$$\{\varphi(x; a_{(i,\alpha(i))}, t_{(i,\alpha(i))}) : i < \omega\} = \{t_{(i,\alpha(i))}(x, a_{(i,\alpha(i))}) : i < \omega\}$$
is consistent, realized by b_α (by 1, above).

□

4 Demonstration of subtlety

In this section, we show that T^*_{feq} does not have the tree property of the first kind (or, in the terminology of Kim and Kim, that it is *subtle*).

Remark 4.1. By Fact 1.7 and Corollary 2.14, it suffices to show that no "complete" formula (that is, a formula that isolates some complete type in its variables) has SOP_2.

Remark 4.2. Any isolating formula $\psi(x_0, \ldots, x_{m-1}, Y_0, \ldots, Y_{n-1}; z_0, \ldots, z_{r-1}, W_0, \ldots, W_{s-1})$ is (equivalent to a formula) of the form

$$\begin{aligned}\psi(xY; zW) := &\bigwedge_{(i,j,k)\in I_1} Y_k(x_i, x_j) \wedge \bigwedge_{(i,j,k)\notin I_1} \neg Y_k(x_i, x_j) \wedge \bigwedge_{(i,j,k)\in I_2} Y_k(x_i, z_j)\\ &\wedge \bigwedge_{(i,j,k)\notin I_2} \neg Y_k(x_i, z_j) \wedge \bigwedge_{(i,j,k)\in I_3} Y_k(z_i, z_j) \wedge \bigwedge_{(i,j,k)\notin I_3} \neg Y_k(z_i, z_j)\\ &\wedge \bigwedge_{(i,j,k)\in J_1} W_k(x_i, x_j) \wedge \bigwedge_{(i,j,k)\notin J_1} \neg W_k(x_i, x_j) \wedge \bigwedge_{(i,j,k)\in J_2} Y_k(x_i, z_j)\\ &\wedge \bigwedge_{(i,j,k)\notin J_2} \neg Y_k(x_i, z_j) \wedge \bigwedge_{(i,j,k)\in J_3} W_k(z_i, z_j) \wedge \bigwedge_{(i,j,k)\notin J_3} \neg W_k(z_i, z_j)\\ &\wedge \bigwedge_{(i,j)\in K_1} x_i = x_j \wedge \bigwedge_{(i,j)\notin K_1} x_i \neq x_j \wedge \bigwedge_{(i,j)\in K_2} Y_i = Y_j \wedge \bigwedge_{(i,j)\notin K_2} Y_i \neq Y_j\\ &\wedge \bigwedge_{(i,j)\in K_3} z_i = z_j \wedge \bigwedge_{(i,j)\notin K_3} z_i \neq z_j \wedge \bigwedge_{(i,j)\in K_4} W_i = W_j \wedge \bigwedge_{(i,j)\notin K_4} W_i \neq W_j\\ &\wedge \bigwedge_{(i,j)\in L_1} x_i = z_j \wedge \bigwedge_{(i,j)\notin L_1} x_i \neq z_j \wedge \bigwedge_{(i,j)\in L_2} Y_i = W_j \wedge \bigwedge_{(i,j)\notin L_2} Y_i \neq W_j\end{aligned}$$

for some $I_1 \subseteq m \times m \times n$, $I_2 \subseteq m \times r \times n$, $I_3 \subseteq r \times r \times n$, $J_1 \subseteq m \times m \times s$, $J_2 \subseteq m \times r \times s$, $J_3 \subseteq r \times r \times s$, $K_1 \subseteq m \times m$, $K_2 \subseteq n \times n$, $K_3 \subseteq r \times r$, $K_4 \subseteq s \times s$, $L_1 \subseteq m \times r$, and $L_2 \subseteq n \times s$. However, we may eliminate the equalities: any equalities among variables of the same 'type' (where by 'type' we mean both sort and either object or parameter - so x_i is the same type as x_j, but not the same type as z_j, for example) simply make one of the variables redundant. That is, since the formula is consistent in the first place, we may replace it by a formula without the redundant variables and the conjuncts containing them. Similarly, suppose there is some equality between an object variable and a parameter variable, say, $x_i = z_j$. Then the realization of x_i is determined by the parameter at z_j. Since the branches of the tree are consistent, the j^{th} point parameter must be the same along any branch, and hence (by Lemma 2.15), throughout the tree. If $\psi(xY; zW)$ has SOP_2, so does the formula ψ' obtained by removing x_i and any conjuncts containing it. (SOP_2 for ψ' is witnessed by the same parameters as for ψ).

Proposition 4.3 (Adapted from [10], Theorem 2.1)**.** *No formula of the form*

$$\begin{aligned}\psi(xY;zW) := &\bigwedge_{(i,j,k)\in I_1} Y_k(x_i,x_j) \wedge \bigwedge_{(i,j,k)\notin I_1} \neg Y_k(x_i,x_j) \wedge \bigwedge_{(i,j,k)\in I_2} Y_k(x_i,z_j)\\ &\wedge \bigwedge_{(i,j,k)\notin I_2} \neg Y_k(x_i,z_j) \wedge \bigwedge_{(i,j,k)\in I_3} Y_k(z_i,z_j) \wedge \bigwedge_{(i,j,k)\notin I_3} \neg Y_k(z_i,z_j)\\ &\wedge \bigwedge_{(i,j,k)\in J_1} W_k(x_i,x_j) \wedge \bigwedge_{(i,j,k)\notin J_1} \neg W_k(x_i,x_j) \wedge \bigwedge_{(i,j,k)\in J_2} Y_k(x_i,z_j)\\ &\wedge \bigwedge_{(i,j,k)\notin J_2} \neg Y_k(x_i,z_j) \wedge \bigwedge_{(i,j,k)\in J_3} W_k(z_i,z_j) \wedge \bigwedge_{(i,j,k)\notin J_3} \neg W_k(z_i,z_j)\\ &\wedge \bigwedge_{i\neq j} x_i \neq x_j \wedge \bigwedge_{i\neq j} Y_i \neq Y_j \wedge \bigwedge_{i\neq j} z_i \neq z_j \wedge \bigwedge_{i\neq j} W_i \neq W_j\\ &\wedge \bigwedge_{i\in \lg(x), j\in\lg(z)} x_i \neq z_j \wedge \bigwedge_{i\in\lg(Y), j\in\lg(W)} Y_i \neq W_j\end{aligned}$$

(where $I_1 \subseteq \lg(x)\times\lg(x)\times\lg(Y)$, $I_2 \subseteq \lg(x)\times\lg(z)\times\lg(Y)$, etc.) has SOP$_2$, and hence, T^*_{feq} is NTP$_1$.

Proof. We work in a sufficiently saturated model $\mathcal{M}$ of T^*_{feq}. (Recall that in Theorem 2.6, we need some degree of saturation to obtain a tree that is 1-fti.) In what follows, we use superscripts to denote coordinates (e.g. a^i is the i^{th} coordinate of a), so as to avoid conflict with subscripts denoting nodes.

Suppose, for a contradiction, that a formula $\psi(xY;zW)$ (of the form listed in the statement of the proposition) has SOP$_2$. By Theorem 2.6, there is a 1-fti tree $\langle a_\alpha r_\alpha : \alpha \in {}^{<\omega}2\rangle$ witnessing SOP$_2$ for $\psi(xY;zW)$ (so $\lg(a_\alpha)=\lg(z)$ and $\lg(r_\alpha)=\lg(W)$ for each α). Note that by Corollary 2.17, since our tree is 1-fti, $a_\eta r_\eta \cap a_\nu r_\nu$ is the same for any $\eta\neq\nu\in {}^{<\omega}2$. In what follows, let $ct := a_{\langle 0\rangle} r_{\langle 0\rangle} \cap a_{\langle 1\rangle} r_{\langle 1\rangle}$ (where $c\in P^{\mathcal{M}}$ and $t\in E^{\mathcal{M}}$). By Remark 2.18, since the tree is 1-fti, $\text{tp}(a_\eta r_\eta/ct) = \text{tp}(a_\nu r_\nu/ct)$ for all $\eta,\nu\in{}^{<\omega}2$. In fact, by the proof of Corollary 2.17, for each $c^i\in c$ and $t^j \in t$ there are k,l such that $a^k_\eta = c^i$ and $r^l_\eta = t^j$ for all η.

Take $b_{\langle 0\rangle}s_{\langle 0\rangle}$ and $b_{\langle 1\rangle}s_{\langle 1\rangle}$ to be realizations of the instances of ψ at the nodes $\langle 0\rangle$ and $\langle 1\rangle$, respectively:

$$b_{\langle 0\rangle}s_{\langle 0\rangle} \models \psi(xY; a_{\langle 0\rangle}r_{\langle 0\rangle});$$
$$b_{\langle 1\rangle}s_{\langle 1\rangle} \models \psi(xY; a_{\langle 1\rangle}r_{\langle 1\rangle}).$$

Let $\mathcal{N}_0$, $\mathcal{N}_1$, and $\mathcal{B}$ be the substructures of $\mathcal{M}$ with universes

$$\begin{aligned} N_0 &= b_{\langle 0\rangle}\cup s_{\langle 0\rangle}\cup a_{\langle 0\rangle}\cup r_{\langle 0\rangle},\\ N_1 &= b_{\langle 1\rangle}\cup s_{\langle 1\rangle}\cup a_{\langle 1\rangle}\cup r_{\langle 1\rangle}, \text{ and}\\ B &= a_{\langle 0\rangle}\cup r_{\langle 0\rangle}\cup a_{\langle 1\rangle}\cup r_{\langle 1\rangle},\end{aligned}$$

respectively (regarding each tuple as the set of its coordinates). Note that these are all models of T_0, and the diagrams of $\mathcal{N}_0$ and $\mathcal{N}_1$ are given by ψ. We shall show that $\mathcal{N}_0$ and $\mathcal{N}_1$ can be amalgamated into a finite model $\mathcal{N}$ of T_0 extending $\mathcal{B}$, with $b_{\langle 0\rangle}s_{\langle 0\rangle}$ and $b_{\langle 1\rangle}s_{\langle 1\rangle}$ being mapped to the same elements of $\mathcal{N}$, $b's'$.

Lemma 4.4. *We can amalgamate $\mathcal{N}_0$ and $\mathcal{N}_1$ into some $\mathcal{N} \vDash T_0$ extending $\mathcal{B}$.*

Proof. We take the universe of $\mathcal{N}$ to be $\{b's', a_{\langle 0\rangle} r_{\langle 0\rangle}, a_{\langle 1\rangle} r_{\langle 1\rangle}\}$ (where b' and s' are new, of lengths $\lg(b_{\langle 0\rangle})$ and $\lg(s_{\langle 0\rangle})$, respectively), and define three maps into $\mathcal{N}$:

$$f_{\mathcal{B}} : \mathcal{B} \hookrightarrow \mathcal{N} \text{ is the identity on } \mathcal{B}$$

$$f_0 : \mathcal{N}_0 \hookrightarrow \mathcal{N} \text{ is given by } \begin{cases} b_{\langle 0\rangle} s_{\langle 0\rangle} & \mapsto b's' \\ a_{\langle 0\rangle} r_{\langle 0\rangle} & \mapsto a_{\langle 0\rangle} r_{\langle 0\rangle} \end{cases}$$

$$f_1 : \mathcal{N}_1 \hookrightarrow \mathcal{N} \text{ is given by } \begin{cases} b_{\langle 1\rangle} s_{\langle 1\rangle} & \mapsto b's' \\ a_{\langle 1\rangle} r_{\langle 1\rangle} & \mapsto a_{\langle 1\rangle} r_{\langle 1\rangle} \end{cases}$$

We claim that we can define R on N so that each of these maps is an embedding, and $\mathcal{N} \vDash T_0$. We begin with $R_0 := f_0(R^{\mathcal{N}_0}) \cup f_1(R^{\mathcal{N}_1}) \cup R^{\mathcal{B}}$.

Claim 4.5. *R_0 adds no "new relations." That is, if $\mathcal{C} \in \{\mathcal{N}_0, \mathcal{N}_1, \mathcal{B}\}$ and $d, e \in \mathcal{C}^P$ and $q \in \mathcal{C}^E$, then $f_{\mathcal{C}}(d, e, q) \in R_0$ if and only if $(d, e, q) \in R^{\mathcal{C}}$. Equivalently, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct structures from the set $\{\mathcal{N}_0, \mathcal{N}_1, \mathcal{B}\}$, and $d, e \in f_{\mathcal{C}_1}(\mathcal{C}_1^P) \cap f_{\mathcal{C}_2}(\mathcal{C}_2^P)$ and $q \in f_{\mathcal{C}_1}(\mathcal{C}_1^E) \cap f_{\mathcal{C}_2}(\mathcal{C}_2^E)$, then $(d, e, q) \in f_{\mathcal{C}_1}(R^{\mathcal{C}_1})$ if and only if $(d, e, q) \in f_{\mathcal{C}_2}(R^{\mathcal{C}_2})$.*

Proof of claim. There are three cases to consider.

- $(d, e, q) \in f_0(N_0) \cap f_1(N_1)$: There are several subcases, depending on whether each of d, e, and q is in B or not, but in all cases the result follows from the fact that $\mathcal{N}_0$ and $\mathcal{N}_1$ have the same diagram. For example, if there are i, j, and k such that $d = b'^i$, $e = b'^j$, and $q = s'^k$, then

$$\begin{aligned}(d, e, q) \in f(R^{\mathcal{N}_0}) &\leftrightarrow \mathcal{N}_0 \vDash s^k_{\langle 0\rangle}(b^i_{\langle 0\rangle}, b^j_{\langle 0\rangle}) \\ &\leftrightarrow \psi(xY; zW) \vdash R(x^i, x^j, Y^k) \\ &\leftrightarrow \mathcal{N}_1 \vDash s^k_{\langle 1\rangle}(b^i_{\langle 1\rangle}, b^j_{\langle 1\rangle}) \\ &\leftrightarrow (d, e, q) \in f(R^{\mathcal{N}_1}).\end{aligned}$$

  The arguments in the other subcases are the same, except that some or all of the relevant variables in the second line may be from z and W, rather than x and Y. (We are using the fact that elements of c and t come from the same coordinates in $a_{\langle 0\rangle}$ as in $a_{\langle 1\rangle}$.)

- $(d, e, q) \in f_0(N_0) \cap f_{\mathcal{B}}(B)$: In this case, the result follows from the facts that $f_0(N_0) \cap f_{\mathcal{B}}(B) = N_0 \cap B$ and that $\mathcal{N}_0$ and $\mathcal{B}$ are both substructures of $\mathcal{M}$, so $R^{\mathcal{N}_0}$ and $R^{\mathcal{B}}$ agree on $N_0 \cap B$. Suppose $d = f(a^i_{\langle 0\rangle}) = a^i_{\langle 0\rangle}$, $e = f(a^j_{\langle 0\rangle}) = a^j_{\langle 0\rangle}$, and $q = f(r^k_{\langle 0\rangle}) = r^k_{\langle 0\rangle}$. Then we have:

$$\begin{aligned}(d, e, q) \in f_0(R^{\mathcal{N}_0}) &\leftrightarrow \mathcal{N}_0 \vDash r^k_{\langle 0\rangle}(a^i_{\langle 0\rangle}, a^j_{\langle 0\rangle}) \\ &\leftrightarrow \mathcal{M} \vDash r^k_{\langle 0\rangle}(a^i_{\langle 0\rangle}, a^j_{\langle 0\rangle}) \\ &\leftrightarrow \mathcal{B} \vDash r^k_{\langle 0\rangle}(a^i_{\langle 0\rangle}, a^j_{\langle 0\rangle}) \\ &\leftrightarrow (d, e, q) \in f_{\mathcal{B}}(R^{\mathcal{B}}).\end{aligned}$$

- $(d, e, q) \in f_1(N_1) \cap f_{\mathcal{B}}(B)$: The argument is the same as in the previous case.

$\square$

We extend R_0 to all of $P^{\mathcal{N}} \times P^{\mathcal{N}} \times E^{\mathcal{N}}$ as follows:

- $R^{\mathcal{N}}(b'^i, a^j_{\langle 0\rangle}, r^k_{\langle 1\rangle})$ and $R^{\mathcal{N}}(a^j_{\langle 0\rangle}, b'^i, r^k_{\langle 1\rangle})$ hold if and only if there is l such that:

$$\mathcal{N}_1 \models R(b^i_{\langle 1\rangle}, a^l_{\langle 1\rangle}, r^k_{\langle 1\rangle}) \text{ and}$$
$$\mathcal{B} \models R(a^j_{\langle 0\rangle}, a^l_{\langle 1\rangle}, r^k_{\langle 1\rangle}).$$

- $R^{\mathcal{N}}(b'^i, a^j_{\langle 1\rangle}, r^k_{\langle 0\rangle})$ and $R^{\mathcal{N}}(a^j_{\langle 1\rangle}, b'^i, r^k_{\langle 0\rangle})$ hold if and only if there is an l such that:

$$\mathcal{N}_0 \models R(b^i_{\langle 0\rangle}, a^l_{\langle 0\rangle}, r^k_{\langle 0\rangle}) \text{ and}$$
$$\mathcal{B} \models R(a^j_{\langle 1\rangle}, a^l_{\langle 0\rangle}, r^k_{\langle 0\rangle}).$$

- $R^{\mathcal{N}}(a^i_{\langle 0\rangle}, a^j_{\langle 1\rangle}, s'^k)$ and $R^{\mathcal{N}}(a^j_{\langle 1\rangle}, a^i_{\langle 0\rangle}, s'^k)$ hold if and only if there is an l such that:

$$\mathcal{N}_0 \models R(a^i_{\langle 0\rangle}, b^l_{\langle 0\rangle}, s^k_{\langle 0\rangle}) \text{ and}$$
$$\mathcal{N}_1 \models R(a^j_{\langle 1\rangle}, b^l_{\langle 1\rangle}, s^k_{\langle 1\rangle}).$$

Claim 4.6. *$R^{\mathcal{N}}$, as defined above, gives an equivalence relation for each $i \in E^{\mathcal{N}}$.*

Proof of claim. The reflexivity of and symmetry of $R^{\mathcal{N}}$ come from the definition (and the fact that $R^{\mathcal{N}_0}, R^{\mathcal{N}_1}$, and $R^{\mathcal{B}}$ are reflexive and symmetric). We must show that $R^{\mathcal{N}}(\cdot, \cdot, q)$ is transitive for all $q \in E^{\mathcal{N}}$.

1. $q = r^l_{\langle 1\rangle} \in r_{\langle 1\rangle}$. $R^{\mathcal{N}}(u, v, r^l_{\langle 1\rangle})$ and $R^{\mathcal{N}}(v, w, r^l_{\langle 1\rangle})$.

   (a) $v \in a_{\langle 1\rangle}$: then $R^{\mathcal{N}}(u, w, r^l_{\langle 1\rangle})$ follows from one of: transitivity of $R^{\mathcal{N}_1}(\cdot, \cdot, r^l_{\langle 1\rangle})$ (if $u, w \in f_1(N_1)$), transitivity of $R^{\mathcal{B}}(\cdot, \cdot, r^l_{\langle 1\rangle})$ (if $u, w \in f_{\mathcal{B}}(B) = B$), or the definition of $R^{\mathcal{N}}$ (if one of u, w comes from b' - i.e., $f_1(N_1) \setminus (f_{\mathcal{B}}(B) \cap f_1(N_1))$ - and the other comes from $a_{\langle 0\rangle}$ - i.e., $f_{\mathcal{B}}(B) \setminus (f_{\mathcal{B}}(B) \cap f_1(N_1))$.

   (b) $v = a^j_{\langle 0\rangle} \in a_{\langle 0\rangle}$.

   - If $u, w \in f_{\mathcal{B}}(B) = B$, then $R^{\mathcal{N}}(u, w, r^l_{\langle 1\rangle})$ follows from transitivity of $R^{\mathcal{B}}(\cdot, \cdot, r^l_{\langle 1\rangle})$. Otherwise, one or both of u, w comes from b'.
   - Suppose $u = b'^i$, $w = b'^k$ (i.e., $u, b \in N \setminus f_{\mathcal{B}}(B)$). Then by the definition of $R^{\mathcal{N}}$, $R^{\mathcal{N}}(b'^i, a^j_{\langle 0\rangle}, r^l_{\langle 1\rangle})$ and $R^{\mathcal{N}}(a^j_{\langle 0\rangle}, b'^k, r^l_{\langle 1\rangle})$ imply that there are m and n such that:

   $$R^{\mathcal{N}_1}(b^i_{\langle 1\rangle}, a^m_{\langle 1\rangle}, r^l_{\langle 1\rangle}) \text{ and } R^{\mathcal{B}}(a^j_{\langle 0\rangle}, a^m_{\langle 1\rangle}, r^l_{\langle 1\rangle})$$
   $$R^{\mathcal{B}}(a^j_{\langle 0\rangle}, a^n_{\langle 1\rangle}, r^l_{\langle 1\rangle}) \text{ and } R^{\mathcal{N}_1}(b^k_{\langle 1\rangle}, a^n_{\langle 1\rangle}, r^l_{\langle 1\rangle})$$

   By transitivity of $R^{\mathcal{B}}(\cdot, \cdot, r^l_{\langle 1\rangle})$, it follows that $R^{\mathcal{B}}(a^m_{\langle 1\rangle}, a^n_{\langle 1\rangle}, r^l_{\langle 1\rangle})$. By Claim 4.5, $R^{\mathcal{N}_1}(a^m_{\langle 1\rangle}, a^n_{\langle 1\rangle}, r^l_{\langle 1\rangle})$ holds, too. Two applications of the transitivity of $R^{\mathcal{N}_1}(\cdot, \cdot, r^l_{\langle 1\rangle})$ give us that $R^{\mathcal{N}_1}(b^i_{\langle 1\rangle}, b^k_{\langle 1\rangle}, r^l_{\langle 1\rangle})$, and thus that $R^{\mathcal{N}}(b'^i, b'^k, r^l_{\langle 1\rangle})$.

- Suppose $u = b'^i$ and $w \in B$. We have $a^m_{\langle 1\rangle}$ as above. Since $R^{\mathcal{B}}(a^j_{\langle 0\rangle}, w, r^l_{\langle 1\rangle})$, transitivity of $R^{\mathcal{B}}(\cdot,\cdot,r^l_{\langle 1\rangle})$ gives us that $R^{\mathcal{B}}(a^m_{\langle 1\rangle}, w, r^l_{\langle 1\rangle})$. Then $R^{\mathcal{N}}(b'^i, w, r^l_{\langle 1\rangle})$ comes from either transitivity of $R^{\mathcal{N}_1}(\cdot,\cdot,r^l_{\langle 1\rangle})$ (if $w \in a_{\langle 1\rangle}$, in which case we use, once again, the fact that $R^{\mathcal{B}}$ and $R^{\mathcal{N}}_1$ agree on elements of $a_{\langle 1\rangle}r_{\langle 1\rangle}$) or from the definition of $R^{\mathcal{N}}$ (if $w \in a_{\langle 0\rangle}$).

(c) $v = b'^j \in b'$. The reasoning here will be the same as in the previous subcase.

2. $q = r^l_{\langle 0\rangle} \in r_{\langle 0\rangle}$. $R^{\mathcal{N}}(u, v, r^l_{\langle 0\rangle})$ and $R^{\mathcal{N}}(v, w, r^l_{\langle 0\rangle})$. The reasoning here will be the same as in the previous case.

3. $q = s'^l \in s'$. $R^{\mathcal{N}}(u, v, s'^l)$ and $R^{\mathcal{N}}(v, w, s'^l)$.

   (a) $v \in b'$. Then $R^{\mathcal{N}}(u, w, s'^l)$ follows from transitivity in $R^{\mathcal{N}_1}$ (respectively, $R^{\mathcal{N}_0}$) if u and w are both in $f_1(N_1)$ (respectively, $f_0(N_0)$). If $u \in a_{\langle 0\rangle}$ and $w \in a_{\langle 1\rangle}$ (or vice versa), $R^{\mathcal{N}}(u, w, s'^l)$ comes from the definition of $R^{\mathcal{N}}$.

   (b) $v = a^j_{\langle 0\rangle} \in a_{\langle 0\rangle}$.

   - If $u, w \in f_0(N_0)$, then $R^{\mathcal{N}}(u, w, s'^l)$ comes from transitivity in $R^{\mathcal{N}_0}$. Otherwise, one or both of u, w comes from $a_{\langle 1\rangle}$.
   - Suppose $u = a^i_{\langle 1\rangle}$, $w = a^k_{\langle 1\rangle}$. Then, by definition of $R^{\mathcal{N}}$, $R^{\mathcal{N}}(a^i_{\langle 1\rangle}, a^j_{\langle 0\rangle}, s'^l)$ and $R^{\mathcal{N}}(a^j_{\langle 0\rangle}, a^k_{\langle 1\rangle}, s'^l)$ imply that there are m and n such that:

$$R^{\mathcal{N}_1}(a^i_{\langle 1\rangle}, b^m_{\langle 1\rangle}, s^l_{\langle 1\rangle}) \text{ and } R^{\mathcal{N}_0}(a^j_{\langle 0\rangle}, b^m_{\langle 0\rangle}, s^l_{\langle 0\rangle})$$
$$R^{\mathcal{N}_0}(a^j_{\langle 0\rangle}, b^n_{\langle 0\rangle}, s^l_{\langle 0\rangle}) \text{ and } R^{\mathcal{N}_1}(a^k_{\langle 1\rangle}, b^n_{\langle 1\rangle}, s^l_{\langle 1\rangle})$$

     By transitivity in $R^{\mathcal{N}_0}$, this gives us that $R^{\mathcal{N}_0}(b^m_{\langle 0\rangle}, b^n_{\langle 0\rangle}, s^l_{\langle 0\rangle})$. Since $R^{\mathcal{N}_0}$ and $R^{\mathcal{N}_1}$ agree on their respective preimages of $b's'$, $R^{\mathcal{N}_1}(b^m_{\langle 1\rangle}, b^n_{\langle 1\rangle}, s^l_{\langle 1\rangle})$ holds. Then by transitivity (twice) in $R^{\mathcal{N}_1}$, we have $R^{\mathcal{N}_1}(a^i_{\langle 1\rangle}, a^k_{\langle 1\rangle}, s^l_{\langle 1\rangle})$, and thus $R^{\mathcal{N}}(a^i_{\langle 1\rangle}, a^k_{\langle 1\rangle}, s'^l)$, as desired.
   - Suppose $u = a^i_{\langle 1\rangle}$ and $w \in f_0(N_0)$. We have b'^m as above. Then, since $R^{\mathcal{N}_0}(a^j_{\langle 0\rangle}, w, s^l_{\langle 0\rangle})$, transitivity in $R^{\mathcal{N}_0}$ gives us $R^{\mathcal{N}_0}(w, b^m_{\langle 0\rangle}, s^l_{\langle 0\rangle})$. Then, by definition of $R^{\mathcal{N}}$ (if $w \in a_{\langle 0\rangle}$) or by transitivity in $R^{\mathcal{N}_1}$ (if $w \in b'$), it follows that $R^{\mathcal{N}}(a^i_{\langle 1\rangle}, w, s'^l)$.

   (c) $v = a^j_{\langle 1\rangle} \in a_{\langle 1\rangle}$. The reasoning here will be the same as in the previous subcase.

□

Claim 4.6 shows that $\mathcal{N} \models T_0$. We now observe that, by Claim 4.5 and the fact that the extension of the definition of R_0 to $P^{\mathcal{N}} \times P^{\mathcal{N}} \times E^{\mathcal{N}}$ adds no new relations among images of elements from the same structure, f_0, f_1, and $f_{\mathcal{B}}$ are embeddings, and $\mathcal{N}$ is the desired amalgam. This finishes the proof of Lemma 4.4. □

Since $f_0 : \mathcal{N}_0 \hookrightarrow \mathcal{N}$ and $f_1 : \mathcal{N}_1 \hookrightarrow \mathcal{N}$, and since ψ is quantifier free,

$$\mathcal{N} \models \psi(b's'; a_{\langle 0\rangle}r_{\langle 0\rangle}) \wedge \psi(b's'; a_{\langle 1\rangle}r_{\langle 1\rangle}).$$

As $\mathcal{N} \models T_0$ and T^*_{feq} is the model completion of T_0, $\mathcal{N}$ embeds into a model $\mathcal{N}^*$ of T^*_{feq}. Again, since ψ is quantifier free,

$$\mathcal{N}^* \models \psi(b's'; a_{\langle 0\rangle} r_{\langle 0\rangle}) \wedge \psi(b's'; a_{\langle 1\rangle} r_{\langle 1\rangle}).$$

Using, once again, the fact that T^*_{feq} is the model completion of T_0, we note that $T^*_{\text{feq}} \cup \text{diag}(\mathcal{B})$ is a complete $\mathcal{L}_B$-theory. Since

$$(\mathcal{N}^*, a_{\langle 0\rangle}, r_{\langle 0\rangle}, a_{\langle 1\rangle}, r_{\langle 1\rangle}) \models T^*_{\text{feq}} \cup \text{diag}(\mathcal{B}),$$

and since $\exists x \exists Y(\psi(xY; a_{\langle 0\rangle} r_{\langle 0\rangle}) \wedge \psi(xY; a_{\langle 1\rangle} r_{\langle 1\rangle}))$ is an $\mathcal{L}_B$-sentence in $\text{Th}(\mathcal{N}^*, a_{\langle 0\rangle}, r_{\langle 0\rangle}, a_{\langle 1\rangle}, r_{\langle 1\rangle})$, $T^*_{\text{feq}} \cup \text{diag}(\mathcal{B}) \vdash \exists x \exists Y(\psi(xY; a_{\langle 0\rangle} r_{\langle 0\rangle}) \wedge \psi(xY; a_{\langle 1\rangle} r_{\langle 1\rangle}))$. We also know that

$$(\mathcal{M}, a_{\langle 0\rangle}, r_{\langle 0\rangle}, a_{\langle 1\rangle}, r_{\langle 1\rangle}) \models T^*_{\text{feq}} \cup \text{diag}(\mathcal{B}),$$

and so

$$\mathcal{M} \models \exists x \exists Y(\psi(xY; a_{\langle 0\rangle} r_{\langle 0\rangle}) \wedge \psi(xY; a_{\langle 1\rangle} r_{\langle 1\rangle})),$$

contradicting the choice of $\langle a_\alpha r_\alpha : \alpha \in {}^{<\omega}2\rangle$ as an SOP_2 tree for $\psi(xY; zW)$. □

Remark 4.7. At first glance, one might worry that the proof of Proposition 4.3 contradicts the fact that T^*_{feq} has the tree property, since a tree witnessing TP must contain nodes $\langle 0\rangle$ and $\langle 1\rangle$ such that the instances of the formula in question corresponding to these nodes are inconsistent with each other. However, Proposition 4.3 relies on the fact that an SOP_2 (or TP_1) tree can be chosen to be 1-fti. The same is not always true for formulae with (only) the tree property, as we noted in Remark 2.8.

References

[1] Artem Chernikov. "Theories without the tree property of the second kind". In: *Annals of Pure and Applied Logic* 165.2 (2014), pp. 695–723.

[2] Mirna Džamonja and Saharon Shelah. "On $\triangleleft^*$-maximality". In: *Annals of Pure and Applied Logic* 125.1-3 (2004), pp. 119–158.

[3] Mirna Džamonja and Saharon Shelah. "On $\triangleleft^*$-maximality". Corrected version of the published paper of the same name; available at http://www.uea.ac.uk// h020/692fix4.pdf. 2011.

[4] Wilfrid Hodges. *A shorter model theory.* Cambridge: Cambridge University Press, 1997, pp. x+310.

[5] Byunghan Kim and Hyeung-Joon Kim. "Notions around tree property 1". In: *Annals of Pure and Applied Logic* 162.9 (2011), pp. 698–709.

[6] Byunghan Kim, Hyeung-Joon Kim, and Lynn Scow. "Tree indiscernibilities, revisited". Preprint; to appear in 'Archive for Mathematical Logic'. 2013.

[7] Lynn Cho Scow. "Characterization Theorems by Generalized Indiscernibles". PhD thesis. University of California, Berkeley, 2010.

[8] Saharon Shelah. *Classification theory and the number of nonisomorphic models.* Vol. 92. Studies in Logic and the Foundations of Mathematics. Amsterdam: North-Holland Publishing Co., 1978, pp. xvi+544.

[9] Saharon Shelah. "Simple unstable theories". In: *Annals of Mathematical Logic* 19.3 (1980), pp. 177–203.

[10] Saharon Shelah and Alexander Usvyatsov. "More on SOP_1 and SOP_2". In: *Annals of Pure and Applied Logic* 155.1 (2008), pp. 16–31.